Keywords: Correlation matrix, Kronecker product, measurements, MIMO channels.

Abstract
Kronecker-separable MIMO channels, especially for larger MIMO configurations, are found more often in system simulations than in reality and some good physical explanations for the latter are available. This paper treats testing Kronecker-separability from a statistical perspective. The point raised is that for larger configurations the dimensions of the full correlation matrix become too large (equal to $[NM \times NM]$, with $N$ number of receivers and $M$ number of transmitters) for reliable estimation with the limited number of realisations normally available. Some examples are given, based on synthetic and measured data.

1 Introduction
Correlation matrices are often used for statistical modelling of MIMO channels. The essence of such modelling is that received signals inherently are stochastic with correlation of any degree between transmitters, between receivers, and between transmit and receive side, as expressed in the channel correlation matrix. From where these correlations originate is deemed irrelevant. Treating the interaction between Tx and Rx is awkward, so, attempts are made to simplify modelling by decoupling the Tx and Rx side and separating the effects on both sides. Probably, the best-known model in this category is the Kronecker model, stating that the transmit- and receive-sided correlation matrix $R_{Tx}$ and $R_{Rx}$ resp.) are fully independent and that the full-correlation matrix can be expressed as the Kronecker-product of both smaller matrices. That means, that, writing the full correlation matrix $R_H$ in the usual way:

$$R_H = \mathbb{E} \{ \text{vec}(H) \text{vec}(H)^H \}$$  \hspace{1cm} (1)

with $H$ the transmission matrix and vec($\cdot$) denoting stacking matrix columns into a single column vector, the Tx-sided correlation matrix $R_{Tx}$ as

$$R_{Tx} = \mathbb{E} \{ H^H H \}$$  \hspace{1cm} (2)

the full correlation matrix of a Kronecker-separable channel can be written as

$$R_H = R_{Tx} \otimes R_{Rx}.$$  \hspace{1cm} (4)

Such a separation is attractive as the full correlation matrix grows rapidly with the size of the antenna systems (the number of elements equals the product of the squares of the numbers of receive and transmit antennas), that a number of off-diagonal elements of the full matrix have no apparent physical interpretation, and that all by all, full correlation matrices are not easily tractable. The consequence of Equation (4) is that the angular spectra at transmit and receive side can be modelled fully independently [1], but, that means that every single scatterer should scatter its power to every other scatterer in the channel as to prevent any coupling between Angle of Departure and Angle of Arrival. It seems that among propagation experts, some form of consensus has been reached that for small MIMO systems like $2 \times 2$, this Kronecker model can hold reasonably well, due to the limited resolution of the small antenna arrays while for larger systems the discrepancies become too large.

For experimental verification, however, the correlation matrices have to be estimated from measured data (that is, from bi-directionally resolved measurements). This is by no means trivial. For a proper estimate of the full correlation matrix of a large MIMO constellation, like $8 \times 16$ as often used in TU Ilmenau measurement campaigns, very large amounts of data samples have to be taken to estimate correlation matrices of size $128 \times 128$, containing over 16,000 matrix elements. Estimation accuracy will be measured by the Correlation Matrix Distance (CMD) in the remainder of this note, which will be discussed in the next section. Additionally, the Diversity Measure will be used also, as this measure has Kronecker-product properties too.

The structure of this paper is as follows, in the next section we will briefly discuss two different metrics for correlation matrices, illustrated with a practical measurement, followed by the influence estimation of correlation matrices has on testing Kronecker separation with another practical example. The last section contains the conclusions.

2 Correlation matrix metrics
In this section, two metrics will be treated briefly that will be used in determining the quality of correlation matrix estimates:
1. the Correlation Matrix Distance (CMD), a metric on differences between correlation matrices, and
2. the Diversity Measure, a metric on eigen value distributions.

2.1 Correlation matrix distance

The Correlation Matrix Distance (CMD) was introduced by Herdin et al. in order to be able to assess the stationarity of mobile MIMO channels. The idea behind it is that, as mentioned, correlation matrices are often used for statistical modelling of MIMO channels. Using the same matrix will result in channels with identical statistical properties and their question was how realistic such an approach is in reality and how fast measured channels change [2,3].

The measure Herdin et al. devised is intended to show changes in the structure of correlation matrices without any restriction to subspaces and independent of received power. The latter mainly hinges on the power-scaling of the transmission matrices to go into the correlation matrices. The Correlation Matrix Distance between two arbitrary correlation matrices \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \) (of identical size) is defined as:

\[
d_{\text{corr}} = 1 - \frac{\text{trace}[\mathbf{R}_1 \mathbf{R}_2^H]}{\|\mathbf{R}_1\|_F \|\mathbf{R}_2\|_F} = 1 - \left\langle \text{vec}(\mathbf{R}_1), \text{vec}(\mathbf{R}_2) \right\rangle
\]

with \( \| \cdot \|_F \) denoting the Frobenius norm and \( \langle \cdot, \cdot \rangle \) the scalar product between two vectors. A CMD value of 0 indicates maximum similarity.

An effect noticed with application of the Correlation Matrix Distance to measurements is that the measure is not insensitive to rotation in case of directional fields, at least not with circular arrays. The point is then whether to appreciate such rotations as a change in the radio channel or in the transmission channel. We define the radio channel as the transmission channel with the influence of the antennas removed. Therefore, the mentioned rotations and resulting changes in the CMD would reflect changes in the transmission channel and not necessarily in the radio channel.

2.2 Diversity Measure

Ivrilac and Nossek introduced the "Diversity Measure" [4] to characterise the fading on multi-antenna systems, effectively a scalar indicating the evenness of the distribution of eigen values of the correlation matrix.

\[
\Psi(\mathbf{R}_H) = \left( \frac{\text{trace}[\mathbf{R}_H]}{\|\mathbf{R}_H\|_F} \right)^2 = \frac{\sum_{i=1}^{K} \lambda_i}{\sum_{i=1}^{K} \lambda_i^2}
\]

The expansion of the second term into the third is by Özelcil et al. [5]. The formulation can be adapted to single-sided correlation matrices, and, in case of "Kronecker-separable" channels, the measure itself is also separable, meaning that if:

\[
\mathbf{R}_H = \mathbf{R}_{\text{Tx}} \otimes \mathbf{R}_{\text{Rx}},
\]

the Diversity Measure \( \Psi(\mathbf{R}_H) \) is directly related to the Diversity Measures on each side:

\[
\Psi(\mathbf{R}_{\text{Tx}}) \cdot \Psi(\mathbf{R}_{\text{Rx}})
\]

However, as mentioned earlier, Kronecker-separable channels are not that common. But, even if the channel is not Kronecker separable, \( \mathbf{R}_{\text{Tx}} \) and \( \mathbf{R}_{\text{Tx}} \) can, in case of rank deficient channels, be used for system adaptation in determining on which end of the link those deficiencies occur.

2.3 Practical measurement

To illustrate how the CMD and Diversity Measure may behave, an example is given for a measurement at Market Square in Ilmenau (Figure 1). The measurement is in LoS at 5.2 GHz. The Tx is mounted on a trolley moving along a straight line approaching the receiving AP, then a sudden change of direction follows, and the measurement proceeds along another straight line, gradually moving out of the receive array beam. Correlation matrices for the Tx and Rx side were made up over each second of the measurement (193 frequency bins spanning 120 MHz and 49 snapshots) and running-averages were determined from these correlation matrices, over an averaging interval of five seconds. Also the Diversity Measure was determined for the same correlation matrices (i.e. averaged over one second). Shown in Figure 2 are both the Correlation Matrix Distance (upper figure) between the actual correlation matrix (1 s average) and its running-average over the last five seconds, for both matrix types, and the Diversity Measures as function of measurement.
time (middle figure). Additionally, the total received power is shown. Apart from a disturbance at the Tx side around nine seconds in the run almost nothing happens up to 35 seconds: the trolley is crossing an empty square in LoS, approaching the AP. Due to the increasing distance to the scattering surfaces of the houses lining the square, the Diversity Measure, already low, drops slightly. At 35 seconds in the run, the trolley takes the bend to proceed on the second straight line, still on the open square. The Tx correlation matrix changes instantly but the amount of scattering does not change and neither does the Diversity Measure. It is not before a lorry blocks LoS (lower figure), the subsequent driving out of the field of view of the receive antenna, and finally leaving the market square, that the scattering becomes more important and the Diversity Measure rises. However, at that moment, the Correlation Matrix Distance settles again at lower values. Note that the change in orientation of the Tx array is also noticed in the Rx correlation matrix, but later, more prolonged and at a lower level. A likely explanation is that after the turn, the receive direction starts to change slowly where it was almost stationary before. Before turning, the trolley with the transmit array was moving almost straightly onto the receive array and the direction of arrival of the dominant component did barely change. After turning, the direction of travel had also a perpendicular component, making the direction of arrival vary over measurement time. The orientation of the Tx array itself is not important at the receive side, simply interchanging Tx columns in the transmission matrix will yield a fully identical Rx correlation matrix.

So, in general, there are circumstances under which a change in the correlation matrix does not reflect a change in statistical properties of the channel. Additionally, changes in the scattering situation are not necessarily indicated clearly in changes in the Correlation Matrix Distance. The Diversity Measure performs much better in this respect.

3 Kronecker separability

3.1 Accuracy vs. number of averages

Correlation matrices may contain very large numbers of elements and thus, when using too few realisations, estimates of correlation matrices may differ appreciably from their expectations. In case of Kronecker separable channels, the variance of the estimate could obscure the separability. As an illustration of this, in Figure 3, Correlation Matrix Distances to respective expectations are shown for correlation matrices derived from $8 \times 16$ channels, for different numbers of realisations used for making-up the correlation matrices, all based on synthetic i.i.d. Rayleigh-fading channels. The values shown are averages over 200 simulations. Additionally, the Correlation Matrix Distance between $\mathbf{R}_H$ and $\mathbf{R}_\text{Tx} \otimes \mathbf{R}_\text{Rx}$ is shown, marked $\text{CMD}_{\text{Kronecker}}$.

The effect is strongest for the correlation matrix $\mathbf{R}_H$ as the ratio of number of realisations and number of matrix elements is by far the lowest, compared to $\mathbf{R}_\text{Tx}$ and $\mathbf{R}_\text{Rx}$. Nevertheless, the effect on the differently sized $\mathbf{R}_\text{Tx}$ and $\mathbf{R}_\text{Rx}$ can still be seen, the former, having four times the number of matrix elements of the latter, needs more samples for the same estimate quality. Note that the CMD values between the full correlation estimate and the Kronecker product.
Figure 3 Effect of number of realisations on estimation of correlation matrices (full $R_{HH}$, $R_{RTx}$, and $R_{RRx}$).

Upper: CMD with Identity Matrices and Kronecker product of $R_{RTx}$ and $R_{RRx}$. Lower: Diversity Measures for the three correlation matrix estimates. Simulated $8 \times 16$ i.i.d. Rayleigh channels.

1. The correlation matrix distances (CMD) are very close to those related to the expectation $I$, suggesting that for a moderate number of samples the Kronecker product itself can be reasonably well estimated from $R_{RTx}$ and $R_{RRx}$. Only $R_{HH}$ can not be estimated accurately enough.

A very similar behaviour is observed for the Diversity Measures in the lower figure. All values are normalised with respect to their theoretical maximum value from Equation (6). For $R_{RTx}$ and $R_{RRx}$, their respective maximum values are reached at approximately the same instance as the CMD with $I$ has become very small. The maximum value of the Diversity Measure for $R_{HH}$, equal to the product of the values for $R_{RTx}$ and $R_{RRx}$, is only reached after taking orders of magnitude larger numbers of realisations.

The relevance for measured channels is that for Rayleigh-fading channels in systems of the size of $8 \times 16$, large numbers of independent realisations are needed to arrive at a good estimate of the full correlation matrix. Essentially the same reasoning holds for Kronecker separability. When a CMD of estimate to expectation of 0.1 would suffice, about 1000 samples are needed. For channel sounding in 120 MHz bandwidth, somewhere between 40 and 100 uncorrelated samples can be had over bandwidth, as a rule of thumb, depending on the correlation bandwidth of the channel. That means that at least ten other snapshots should be measured, each separated by the coherence-time or length of the channel or more. Over those multiple coherence lengths, the channel should remain stationary.

3.2 Testing Kronecker separability for a practical example

For the measurement run discussed in Section 2.3, Kronecker separability was tested. However, as shown in the previous section, the $8 \times 16$ constellation of the measurements requires very many independent realisations. Therefore, the measurement constellation was, during data processing, artificially reduced to $4 \times 4$. From Figure 3, a $16 \times 16$ correlation matrix (the one resulting from a $4 \times 4$ MIMO system) can adequately be estimated with from 50 realisations upwards.

Actually, two different selections were made, a "LoS" selection and a "dual-polarised Obstructed LoS". The first consists of the four central patch elements of the uniform linear receive array and the four antenna elements of the uniform circular Tx array on the measurement trolley that were facing the receive array on the first part of the run. The transmit array is nominally vertically polarised and of the receive array only the "vertical" designated ports were used. Note that the antenna element types used here have considerable sensitivity for the other polarisation direction, especially for angles out of the zero-elevation plane. For the second selection, only the two centre patches of the linear receive array were used, but both ports of each element (the "Vertical" and "Horizontal" port, respectively) were selected. It is expected that the Kronecker separability will be degraded...
as the cross-polarisation, either originating from the Tx antenna elements, from scatterers, or from the Rx antenna elements, makes it less likely that the angular receive spectrum is independent of the transmit antenna element or the transmit spectrum independent of the receive element.

The Obstructed LoS refers to selecting the four Tx antenna elements that showed the largest transmission losses to the receive array on the first part of the measurement run. As the transmit array has an absorptive core to reduce coupling between elements, the transmission loss to the receiver from the elements not facing the receiver is larger than in LoS. The objective of selecting these elements is making multi-path contributions more important. The increased transmission loss can be appreciated from the lower part of Figure 6. The Diversity Measure for both types of correlation matrices (not shown) has a value of around 1 for the LoS selection, rising to about 2 to 3 after $t=45$ s, indicating a rank 1 channel initially with increased scattering afterwards. The Diversity Measure for the OLoS selection has initially values around 2 and drops to somewhat lower values after $t=35$ s.

In the LoS case, the total received power is lower than in Figure 2, which is caused by halving the number of Rx elements and reducing the number of effectively contributing Tx elements. In the other case, the cross-polarisation incurs an additional loss on top of that by the obstruction. The rotation of the measurement trolley around $t = 35$ s mentioned in Section 2.3 is noticeable in the faster increase of received power with time for the OLoS selection.

The upper and middle part of Figure 6 show the CMD between the full correlation matrix and the Kronecker product of Tx and Rx-sided correlation matrices. The solid (blue) lines with markers represent averages over 5 seconds, the solid (black) lines are from averages over one second. The top part shows results for the dual-polarised Rx/Obstructed LoS selection, the middle part those for the LoS selection. It will be clear that the best match is for the (almost trivial) LoS selection, in an almost rank one channel that is separable. Part of the difference is made up after $t = 35$ s, when the CMD values for the OLoS selection drop as the Tx antenna elements are slowly rotated into LoS.

4 Conclusions

Proper estimation of correlation matrices is vital for assessing Kronecker-separability of MIMO channels. Especially for larger MIMO configurations, the amount of independent data needed for such estimations could exceed what is practically available from measurements. The trivial case of LoS will likely be well separable.

Acknowledgements

This work was performed in the context of the Newcom FP6 Network of Excellence.

References

Figure 6: Correlation matrix distance between full correlation matrix $R_{hh}$ and Kronecker product of $R_{tx}$ and $R_{rx}$ on Ilmenau market square, for two $4\times4$ configurations, dual polarised access point and shadowed Tx antennas (top) and LoS $4\times4$ single polarised V-V (bottom). Two different averaging windows, 1 s and 5 s respectively.